Fermi density distribution for the nucleus. 31 July 2005

Usually we use the model of uniformly charged sphere (UCS). Now we want to estimate possible errors of this model for the operators, which are sensitive to the nuclear distribution. To do this we compare UCS model with Fermi model. Vasilii Korol started this work and here I try to reproduce his results.

> `restart;`

`assume(a::real) assume(c::real)`

Fermi distribution is given by the function (not normalized): `f := r -> 1/(1+exp((r-c)/a))`

$$f := r \rightarrow \frac{1}{1 + e^{\left(\frac{r-c}{a}\right)}}$$

For small a this function is close to $\theta(c - r)$. Let us expand it in $\frac{a}{c}$.

> `d1:=diff(f(r),a);`

$$d1 := \frac{(r - c)\, e^{\left(\frac{r-c}{a}\right)}}{(1 + e^{\left(\frac{r-c}{a}\right)})^2\, a^2}$$
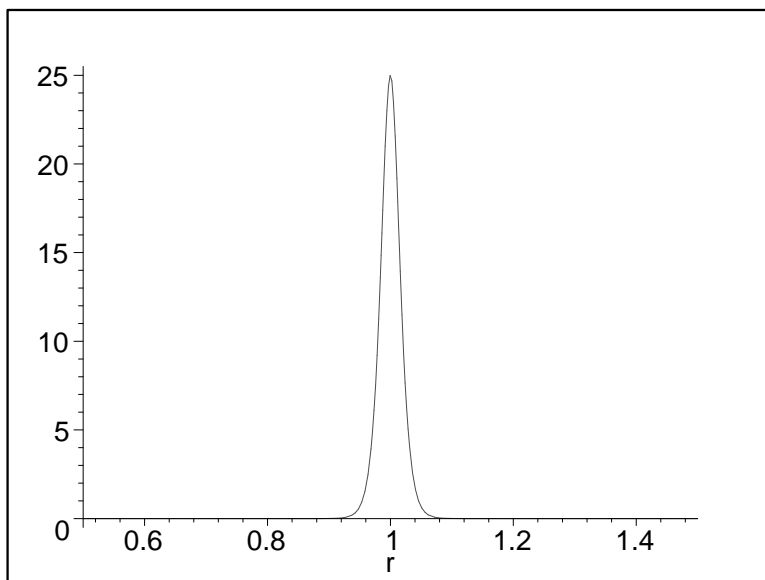
For a=0 this expression is zero for $r < c$ and for $r > c$. Let us see how it looks near $r = c$. Consider first :

> `d2:=(d1*a/(r-c));`

$$d2 := \frac{e^{\left(\frac{r-c}{a}\right)}}{(1 + e^{\left(\frac{r-c}{a}\right)})^2\, a}$$

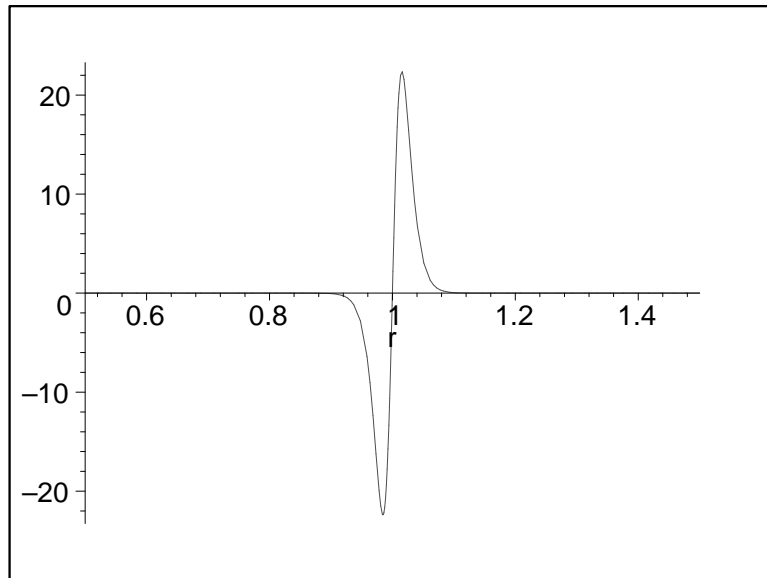Let us plot d2 and d1 for `c:=1:  a:=0.01:`

> `plot(d2, r=0.5..1.5);`

```
>   int(d2, r=0.5..1.5);
```
$$1.000000000$$

So, d2 behaves as delta-function. Then, d1 should be linked to $\frac{\partial}{\partial r}\,\delta(r-c)$.

```
>   plot(d1, r=0.5..1.5);
```



```
>   int(3*d1*(r-c)/(Pi^2*a), r=0.5..1.5);
```
$$1.000000000$$

So, we introduced extra factor $\frac{3}{\pi^2\,a}$ to get rifht normalization.

We see that $d2$ goes to $\delta(r-c)$ and $\frac{3\,d1}{\pi^2\,a} = -\frac{d\,\delta(r-c)}{dr}$. That means that f(r) in fact does not have first derivative in a and expansion starts with a^2:

$$\boxed{f(r) = \theta(c-r) - \frac{\pi^2\,a^2}{6}\,\frac{\partial}{\partial r}\,\delta(r-c) + \ldots}$$

Now we can evaluate $\langle r^2 \rangle$. First, we have to calculate normalization:

$$V = \int_0^\infty f(r)\,r^2\,dr = \frac{c^3}{3} + \frac{\pi^2\,c\,a^2}{3}.$$

Now $V\,\langle r^2 \rangle = \int_0^\infty f(r)\,r^4\,dr = \frac{c^5}{5} + \frac{2\,\pi^2\,c^3\,a^2}{3}.$

From this we get rms radius:

$$\boxed{\langle r^2 \rangle = \frac{3\,c^2}{5} + \frac{7\,\pi^2\,a^2}{5}}$$

Correction in the expression above agrees with Eq.(4) in Korol's notes.

Still, let us check it numerically.

```
>  with(student, simpson);
```
$$[simpson]$$

```
>  c:=1; a:=0.03; simpson(f(r)*r^2, r=0..3, 200);
```
$$c := 1$$

$$a := 0.03$$

$$0.5014724240\,10^{-30} + \frac{1}{50}\left(\sum_{i=1}^{100}\frac{(\frac{3\,i}{100}-\frac{3}{200})^2}{1+e^{(0.9999999999\,i-33.83333333)}}\right)$$
$$+ \frac{1}{100}\left(\sum_{i=1}^{99}(\frac{9}{10000}\frac{i^2}{1+e^{(0.9999999999\,i-33.33333333)}})\right)$$

```
>  V1:= evalf(%);
```
$$V1 := 0.3362942144$$

```
>  simpson(f(r)*r^4, r=0..3, 200):  V2:=evalf(%);
```
$$V2 := 0.2059585831$$

```
>  r2:=V2/V1;
```
$$r2 := 0.6124357015$$

We can compare V1, V2, and r2 with the above analytical expressions.
```
evalf((V1-1/3)*3/(a*Pi)^2) evalf((V2-1/5)*3/(a*Pi)^2)
evalf((r2-3/5)*5/(a*Pi)^2)
```
$$0.9999999252$$

$$2.012435641$$

$$6.999999971$$

So, we see that correct factors are 1, 2, and 7 in agreement with above equations.

Note also, that factors 1 and 7 are much more stable then factor 2. Probably, the next order cancels for them.